

MAXIMAL AREA INTEGRAL PROBLEM FOR CERTAIN CLASS OF UNIVALENT ANALYTIC FUNCTIONS

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ABSTRACT. One of the classical problems concerns the class of analytic functions f on the open unit disk $|z| < 1$ which have finite Dirichlet integral $\Delta(1, f)$, where

$$\Delta(r, f) = \iint_{|z| < r} |f'(z)|^2 dx dy \quad (0 < r \leq 1).$$

The class $\mathcal{S}^*(A, B)$ of normalized functions f analytic in $|z| < 1$ and satisfies the subordination condition $zf'(z)/f(z) \prec (1 + Az)/(1 + Bz)$ in $|z| < 1$ and for some $-1 \leq B \leq 0$, $A \in \mathbb{C}$ with $A \neq B$, has been studied extensively. In this paper, we solve the extremal problem of determining the value of

$$\max_{f \in \mathcal{S}^*(A, B)} \Delta(r, z/f)$$

as a function of r . This settles the question raised by Ponnusamy and Wirths in [11]. One of the particular cases includes solution to a conjecture of Yamashita which was settled recently by Obradović et. al [9].

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1. INTRODUCTION AND PRELIMINARIES

Let f and g be two analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We say that f is *subordinate* to g , written as $f \prec g$, if there exists an analytic function $w : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ with $w(0) = 0$ such that $f(z) = g(w(z))$ for $z \in \mathbb{D}$. In particular, if g is univalent in \mathbb{D} , then $f \prec g$ is equivalent to $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$; see [3, 4].

Denote by \mathcal{A} the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in \mathbb{D} . The class of univalent functions in \mathcal{A} is denoted by \mathcal{S} . Two subclasses of \mathcal{S} to which we will make frequent reference are \mathcal{S}^* and \mathcal{C} , the subclasses of starlike functions (with respect to the origin) and convex functions, respectively. Recall that a function $f \in \mathcal{S}^*$ is characterized by the condition

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

There are a number of ways the class \mathcal{S}^* has been generalized in the literature and one such generalization is defined as follows: For $-1 \leq B \leq 0$ and $A \in \mathbb{C}$, $A \neq B$, define

$$\mathcal{S}^*(A, B) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{D} \right\}.$$

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The function $k_{A,B}$ defined by

$$(1.1) \quad k_{A,B}(z) := \begin{cases} ze^{Az} & \text{if } B = 0, \\ z(1 + Bz)^{(A/B)-1} & \text{if } B \neq 0 \end{cases}$$

is in $\mathcal{S}^*(A, B)$ and acts the role of extremal function for the class $\mathcal{S}^*(A, B)$. However, if $A = e^{i\alpha}(e^{i\alpha} - 2\beta \cos \alpha)$ with $\beta < 1$ and $B = -1$, then $\mathcal{S}^*(A, B)$ reduces to the class $\mathcal{S}_\alpha(\beta)$ of functions f (called α -spirallike of order β) satisfying the condition

$$\operatorname{Re} \left(e^{-i\alpha} \frac{zf'(z)}{f(z)} \right) > \beta \cos \alpha, \quad z \in \mathbb{D},$$

and recall that each function in $\mathcal{S}_\alpha(\beta)$ is univalent if $\beta \in [0, 1)$ and $\alpha \in (-\pi/2, \pi/2)$ (see [6]). Clearly, $\mathcal{S}_\alpha(\beta) \subset \mathcal{S}_\alpha(0)$ whenever $0 \leq \beta < 1$. Functions in $\mathcal{S}_\alpha(0)$ are called α -spirallike. The class $\mathcal{S}_\alpha(0)$ was introduced by Špaček [20] and the set $\mathcal{Sp} = \cup\{\mathcal{S}_\alpha(0) : \alpha \in (-\pi/2, \pi/2)\}$ is referred to us the class of spirallike functions. As remarked in [6], spirallike functions have been used to obtain important counter-examples in geometric function theory (see also [3, p. 72 and Theorem 8.11]).

More often, the class $\mathcal{S}^*(A, B)$ is studied with the restriction $-1 \leq B < A \leq 1$ (see Janowski [5]) so that the values of $zf'(z)/f(z)$ lie inside the disk in the right half plane with center $(1 - ABr^2)/(1 - B^2r^2)$ and radius $(A - B)r/(1 - B^2r^2)$, and so, the class $\mathcal{S}^*(A, B)$ becomes a subclass of \mathcal{S}^* whenever $-1 \leq B < A \leq 1$. We here list down in Table 1 the certain basic subclasses of the class \mathcal{S}^* that are studied for various choices of the pair (A, B) . Set for an abbreviation $p(z) := zf'(z)/f(z)$.

Year	Authors	$\mathcal{S}^*(A, B)$	Conditions	Subordination form
1921	Nevanlinna [7]	$\mathcal{S}^*(1, -1) = \mathcal{S}^*$	$\operatorname{Re} p(z) > 0$	$p(z) \prec \frac{1+z}{1-z}$
1936	Robertson [13]	$\mathcal{S}^*(1 - 2\beta, -1)$ $= \mathcal{S}^*(\beta), \beta \in [0, 1)$	$\operatorname{Re} p(z) > \beta$	$p(z) \prec \frac{1 + (1 - 2\beta)z}{1 - z}$
1968	Singh [17]	$\mathcal{S}^*(1, 0)$	$ p(z) - 1 < 1$	$p(z) \prec 1 + z$
1968	Padamanabhan [10]	$\mathcal{S}^*(\alpha, -\alpha) := \mathcal{T}(\alpha),$ $\alpha \in (0, 1]$	$\left \frac{p(z) - 1}{p(z) + 1} \right < \alpha$	$p(z) \prec \frac{1 + \alpha z}{1 - \alpha z}$
1974	Singh and Singh [18]	$\mathcal{S}^*(1, \frac{1}{\alpha} - 1), \alpha \geq \frac{1}{2}$	$ p(z) - \alpha < \alpha$	$p(z) \prec \frac{1 + z}{1 + \frac{1-\alpha}{\alpha}z}$
1978	Silverman [16]	$\mathcal{S}^*\left(\frac{b^2 - a^2 + a}{b}, \frac{1-a}{b}\right),$ $a + b \geq 1$ & $a \in [b, 1 + b]$	$ p(z) - a < b$	$p(z) \prec \frac{1 + \frac{b^2 - a^2 + a}{b}z}{1 + \frac{1-a}{b}z}$
2014	Sahoo and Sharma [19]	$\mathcal{S}^*((1 - 2\beta)\alpha, -\alpha)$ $:= \mathcal{T}(\alpha, \beta),$ $\alpha \in (0, 1], \beta \in [0, 1)$	$\left \frac{p(z) - 1}{p(z) + 1 - 2\beta} \right < \alpha$	$p(z) \prec \frac{1 + (1 - 2\beta)\alpha z}{1 - \alpha z}$

Table 1

From (1.1), we note that

$$(1.2) \quad k_{1-2\beta, -1}(z) = \frac{z}{(1-z)^{2(1-\beta)}} := k_\beta(z), \text{ and } k_{\alpha, -\alpha}(z) = \frac{z}{(1-\alpha z)^2}.$$

Suppose that f is a function analytic in \mathbb{D} . We denote by $\Delta(r, f)$, the area of the multi-sheeted image of the disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ ($0 < r \leq 1$) under f . Thus, in terms of the coefficients of f , $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, one gets with the help of the classical Parseval-Gutzmer formula (see [19]) the relation

$$(1.3) \quad \Delta(r, f) = \iint_{\mathbb{D}_r} |f'(z)|^2 dx dy = \pi \sum_{n=1}^{\infty} n |a_n|^2 r^{2n}$$

which is called the Dirichlet integral of f . Computing this area is known as the *area problem for the functions of type f* . Thus a function has a finite Dirichlet integral exactly when its image has finite area (counting multiplicities). All polynomials and, more generally, all functions $f \in \mathcal{A}$ for which f' is bounded on \mathbb{D} are Dirichlet finite.

Our work in this paper is motivated by the work of Yamashita [21] and recent works [8, 9, 11, 19]. In 1990, Yamashita [21] conjectured that

$$(1.4) \quad \max_{f \in \mathcal{C}} \Delta\left(r, \frac{z}{f}\right) = \pi r^2$$

for each r , $0 < r \leq 1$. The maximum is attained only by the rotations of the function $l(z) = z/(1-z)$. In 2013, the Yamashita conjecture was settled in [8] (see Corollary 2.4) in a more general setting including functions from $\mathcal{S}_\alpha(\beta)$ (see [11]). In the recent paper [19], the maximum area problem for the functions of type $z/f(z)$ when $f \in \mathcal{S}^*((1-2\beta)\alpha, -\alpha) \equiv \mathcal{T}(\alpha, \beta)$ ($0 < \alpha \leq 1$, $0 \leq \beta < 1$), is established (see Corollary 2.7).

A general problem on the Yamashita conjecture for the class $\mathcal{S}^*(A, B)$ was suggested in [11] (see also [8, 9]), and partially it is solved in [19]. In this paper, we solve the problem in complete generality for the full class $\mathcal{S}^*(A, B)$, and the main results are stated in Section 2.

2. MAIN THEOREMS

For complex numbers a, b and c with c neither zero nor a negative integer, the Gaussian hypergeometric series ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1.$$

Clearly, the function ${}_2F_1(a, b; c; z)$ is analytic in \mathbb{D} and thus, the shifted function $z {}_2F_1(a, b; c; z)$ belongs to \mathcal{A} . Here $(a)_n$ denotes the shifted factorial notation defined, in terms of the Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1) \cdots (a+n-1) & \text{if } n \geq 1; \\ 1 & \text{if } n = 0, a \neq 0. \end{cases}$$

The asymptotic behaviour of $F(a, b; c; z)$ near $z = 1$ reveals that

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty \quad \text{for } \operatorname{Re} c > \operatorname{Re}(a+b).$$

If either (or both) of a and b is (are) zero or a negative integer(s), then the power series reduces to a polynomial; see [12]. Similarly, ${}_0F_1(a; z)$ is defined as

$${}_0F_1(a; z) = \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!}, \quad |z| < 1.$$

We now state our main results and their proofs will be given in Section 4.

Theorem 2.1. *Let $f(z) \in \mathcal{S}^*(A, 0)$ for $0 < |A| \leq 1$. Then for r , $0 < r \leq 1$, we have*

$$\max_{f \in \mathcal{S}^*(A, 0)} \Delta \left(r, \frac{z}{f} \right) = E_{A,0}(r)$$

where $E_{A,0}(r) = \pi |A|^2 r^2 {}_0F_1(2; |A|^2 r^2)$. The maximum is attained by the rotations of the function $k_{A,0}(z) = ze^{Az}$.

If $A = 1$ in Theorem 2.1, then we get

Corollary 2.2. *Let $f \in \mathcal{S}^*(1, 0)$. Then we have*

$$\max_{f \in \mathcal{S}^*(1, 0)} \Delta \left(r, \frac{z}{f} \right) = \pi r^2 {}_0F_1(2; r^2) \quad \text{for } 0 < r \leq 1,$$

where the maximum is attained only by the rotation of the function $k_{1,0}(z) = ze^z$.

Theorem 2.3. *Let $f \in \mathcal{S}^*(A, B)$ for $-1 \leq B < 0$ and $A \neq B$ and $z/f(z)$ be a non-vanishing analytic function in \mathbb{D} . Then, for $0 < r \leq 1$, we have*

$$\max_{f \in \mathcal{S}^*(A, B)} \Delta \left(r, \frac{z}{f} \right) = E_{A,B}(r) = \pi |\bar{A} - B|^2 r^2 {}_2F_1(A/B, \bar{A}/B; 2; B^2 r^2).$$

The maximum is attained for the rotations of the function $k_{A,B}(z)$ defined by (1.1).

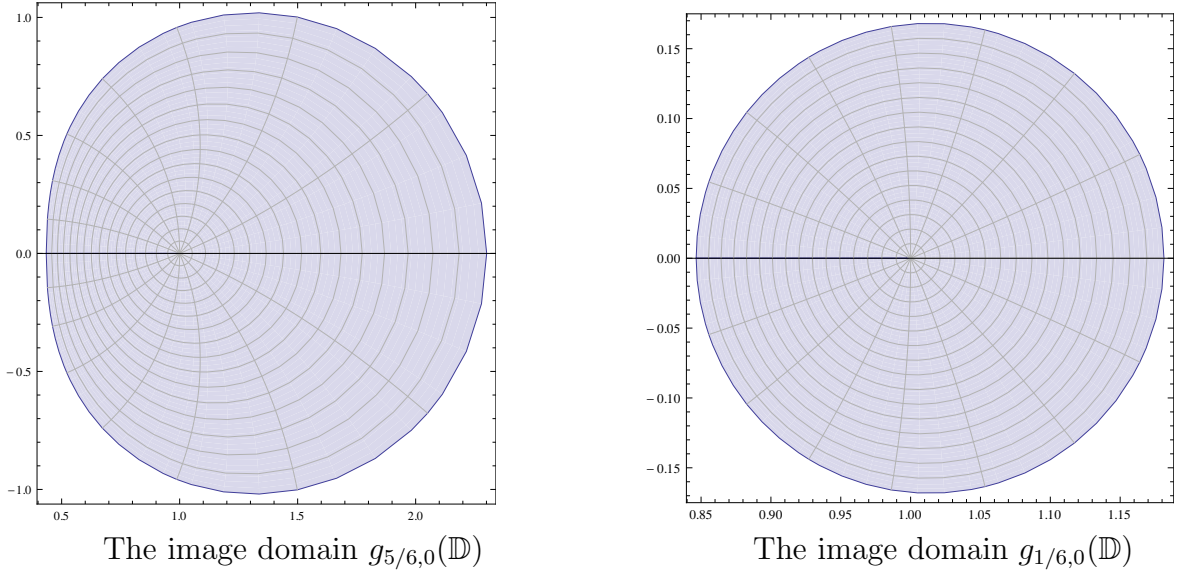
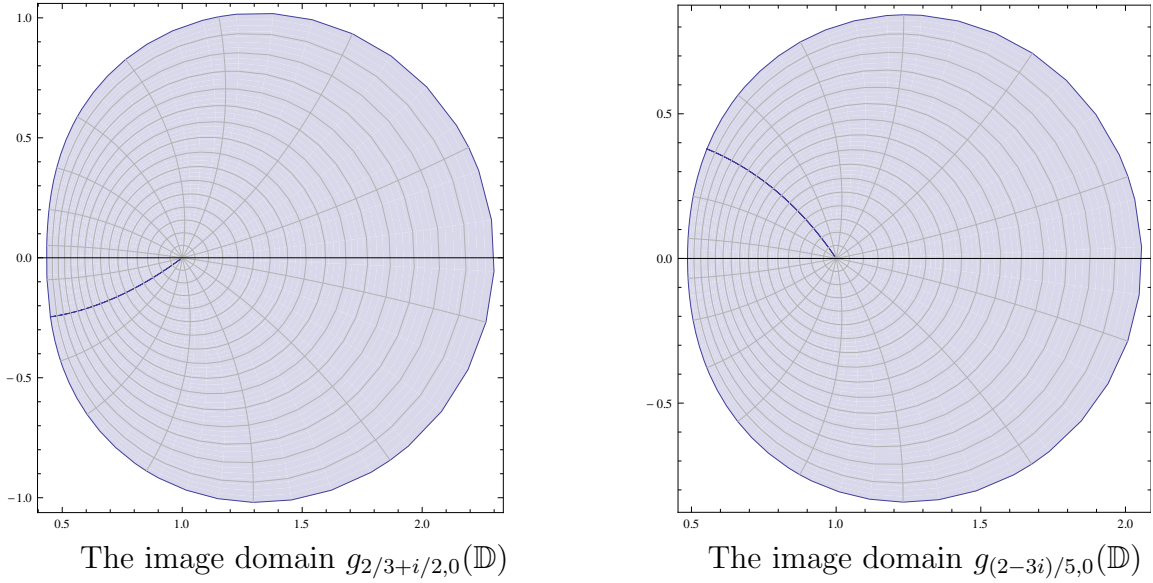
Note that Theorems 2.1 and 2.3 generalize the results proved in [8, 19, 21]. To see the bounds for the Dirichlet finite function, we write

$$E_{A,0}(1) = \pi |A|^2 \sum_{n=0}^{\infty} \frac{1}{(1)_n (2)_n} |A|^{2n} \quad \text{and} \quad E_{A,B}(1) = \pi |\bar{A} - B|^2 \sum_{n=0}^{\infty} \frac{(A/B)_n (\bar{A}/B)_n}{(1)_n (2)_n} B^{2n}.$$

For certain values of A and B , the images of the unit disk under the extremal functions $g_{A,0}(z) = z/k_{A,0}(z) = e^{-Az}$ and $g_{A,B}(z) = z/k_{A,B}(z) = (1 + Bz)^{1-A/B}$, and numerical values of $E_{A,0}(1)$ and $E_{A,B}(1)$ are described in Figures 1–4 and Table 2, respectively. We remind the reader that for $B = -1$, $E_{A,B}(1)$ is finite only if $2 > \operatorname{Re}((A + \bar{A})/B)$, i.e. if $\operatorname{Re} A > -1$.

A	Approximate Values of $E_{A,0}(1)$	B	Approximate Values of $E_{A,B}(1)$
5/6	3.03211	-4/5	11.2917
1/6	0.0884841	-1	4.34607
$2/3 + i/2$	3.03211	-1/2	6.90284
$(2 - 3i)/5$	2.09682	-3/5	5.4645

Table 2

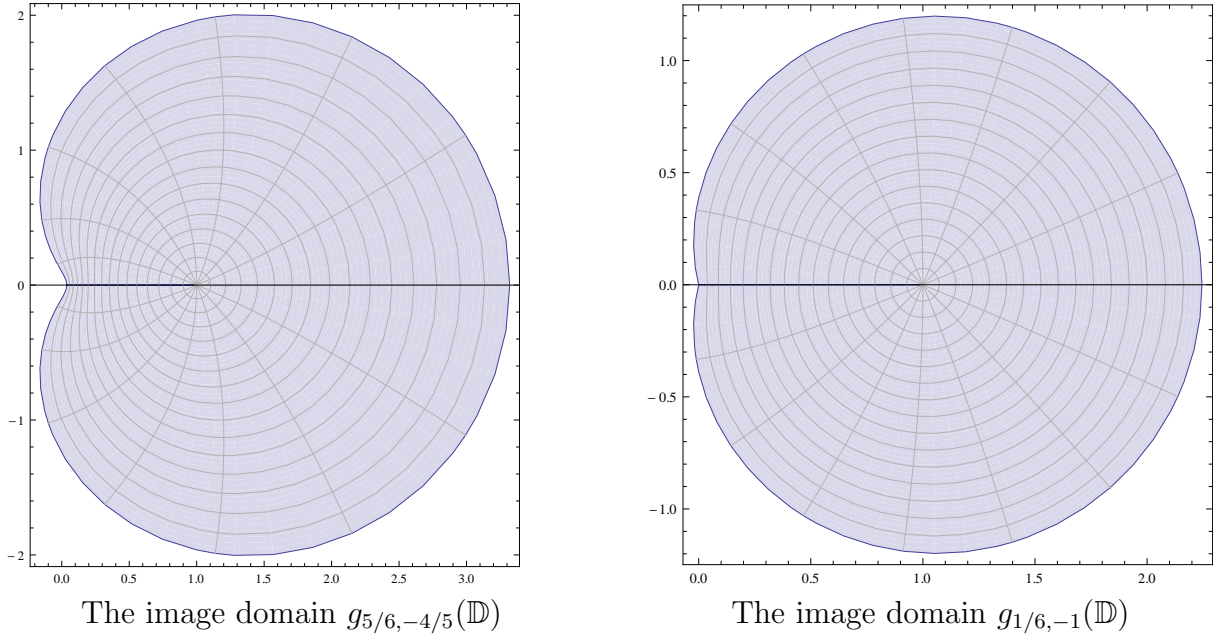
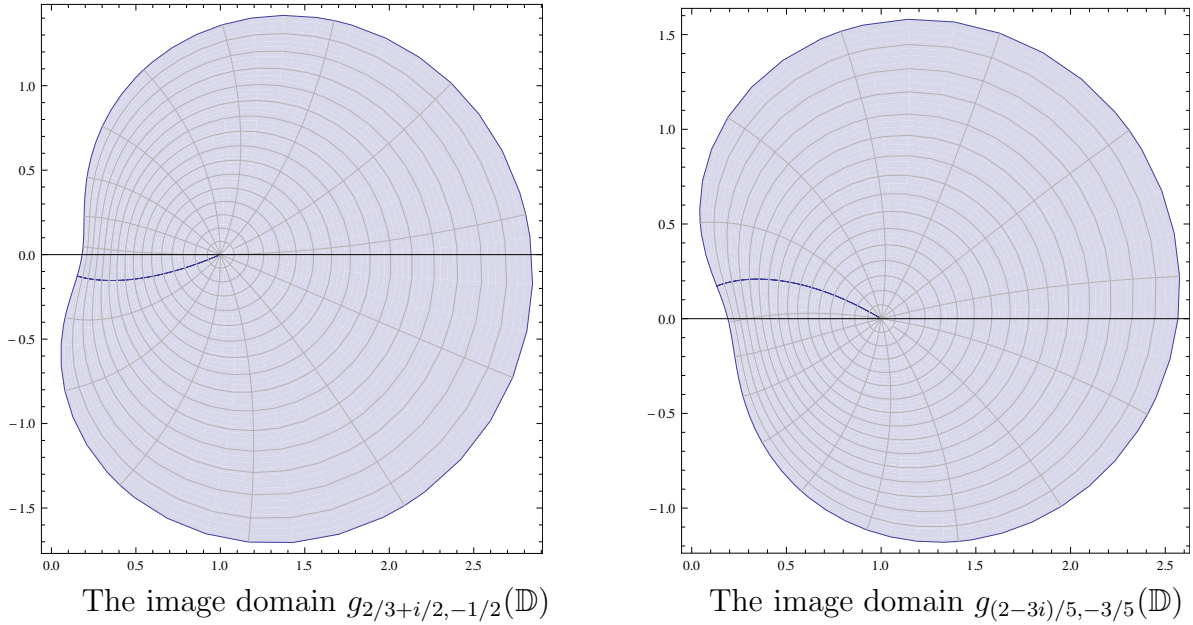
FIGURE 1. Images of the unit disk under $g_{5/6,0}$ and $g_{1/6,0}$.FIGURE 2. Images of the unit disk under $g_{2/3+i/2,0}$ and $g_{(2-3i)/5,0}$.

We now state certain consequences of Theorem 2.3 for the several of the special classes introduced by several authors (refer Table 1). It is a simple exercise to see that Möbius transformation $w = \phi(z)$ defined by

$$w = \phi(z) = \frac{1 + Az}{1 + Bz}$$

maps the unit disk \mathbb{D} onto the half-plane

$$\operatorname{Re}((1 + \overline{A})w) > \frac{1 - |A|^2}{2}$$

FIGURE 3. Images of the unit disk under $g_{5/6, -4/5}$ and $g_{1/6, -1}$.FIGURE 4. Images of the unit disk under $g_{2/3+i/2, -1/2}$ and $g_{(2-3i)/5, -3/5}$.

whenever $B = -1$ and $A \neq 1$. In particular, if $A = e^{i\alpha}(e^{i\alpha} - 2\beta \cos \alpha)$ ($\beta < 1$), then as remarked in the introduction, the last condition reduces to

$$\operatorname{Re}(e^{-i\alpha}w) > \beta \cos \alpha.$$

If $-1 < B \leq 0$ and $A \neq B$, then ϕ maps \mathbb{D} onto the disk

$$\left| w - \frac{1 - \overline{A}B}{1 - B^2} \right| < \frac{|A - B|}{1 - B^2}.$$

This observation helps us to formulate important special cases.

If we choose $A = 1 - 2\beta$ and $B = -1$ in Theorem 2.3, we get

Corollary 2.4. [8, Theorem 3] *Let $f \in \mathcal{S}^*(\beta)$ for some $0 \leq \beta < 1$. Then we have*

$$\max_{f \in \mathcal{S}^*(\beta)} \Delta \left(r, \frac{z}{f} \right) = 4\pi(1 - \beta)^2 r^2 {}_2F_1(2\beta - 1, 2\beta - 1; 2; r^2) \quad \text{for } 0 < r \leq 1,$$

where the maximum is attained by the rotation of the function $k_\beta(z) = \frac{z}{(1-z)^{2(1-\beta)}}$.

We remark that when $A = 1$ and $B = -1$, Theorem 2.3 turns into [8, Theorem A]. If $A = (1 - 2\beta)\alpha$ and $B = -\alpha$ in Theorem 2.3, then we get

Corollary 2.5. [19, Theorem 1.3] *Let $f \in \mathcal{S}^*((1 - 2\beta)\alpha, -\alpha) = \mathcal{T}(\alpha, \beta)$ for $0 < \alpha \leq 1$ and $0 \leq \beta < 1$. Then we have*

$$\max_{f \in \mathcal{T}(\alpha, \beta)} \Delta \left(r, \frac{z}{f} \right) = 4\pi\alpha^2(1 - \beta)^2 r^2 {}_2F_1(2\beta - 1, 2\beta - 1; 2; \alpha^2 r^2), \quad |z| < r$$

for all r , $0 < r \leq 1$. The maximum is attained by the rotation of the function $k_{(1-2\beta)\alpha, -\alpha}(z)$ defined by (1.1).

The case $\beta = 0$ of Corollary 2.5 (i.e. $A = \alpha$ and $B = -\alpha$ of Theorem 2.3) gives

Example 2.6. [19, Theorem 3.1] If $f \in \mathcal{S}^*(\alpha, -\alpha) := \mathcal{T}(\alpha)$ for some $0 < \alpha \leq 1$, then one has

$$\max_{f \in \mathcal{T}(\alpha)} \Delta \left(r, \frac{z}{f} \right) = 2\pi\alpha^2 r^2 (2 + \alpha^2 r^2)$$

for all r , $0 < r \leq 1$, and the maximum is attained by the rotation of the function $k_{\alpha, -\alpha}(z) = z/(1 - \alpha z)^2$.

If we choose $A = e^{i\alpha}(e^{i\alpha} - 2\beta \cos \alpha)$ and $B = -1$ in Theorem 2.3, we get [11, Theorem 3].

If $A = 1$ and $B = (1 - \alpha)/\alpha$, $\alpha \geq 1/2$, then Theorem 2.3 yields

Corollary 2.7. *If $\alpha \geq 1/2$, $f \in \mathcal{S}^*(1, (1 - \alpha)/\alpha)$, then we have*

$$\max_{f \in \mathcal{S}^*(1, (1-\alpha)/\alpha)} \Delta \left(r, \frac{z}{f} \right) = \pi \left(2 - \frac{1}{\alpha} \right)^2 r^2 {}_2F_1 \left(\frac{\alpha}{1 - \alpha}, \frac{\alpha}{1 - \alpha}; 2; \left(\frac{\alpha - 1}{\alpha} \right)^2 r^2 \right),$$

for $0 < r \leq 1$, where the maximum is attained by the rotation of the function $k_{1, (1-\alpha)/\alpha}(z)$ defined by (1.1).

If $A = (b^2 - a^2 + a)/b$ and $B = (1 - a)/b$ with $a + b \geq 1$, $a \in [b, 1 + b]$, as a consequence of Theorem 2.3 we obtain the following maxima area problem for functions in a class introduced by Silverman (see Table 1 for the reference).

Corollary 2.8. *Let $f \in \mathcal{S}^*((b^2 - a^2 + a)/b, ((1 - a)/b))$. Then we have*

$$\max_{f \in \mathcal{S}^*((b^2 - a^2 + a)/b, ((1-a)/b))} \Delta \left(r, \frac{z}{f} \right) = \pi s_1^2 r^2 {}_2F_1 \left(s_2, s_2; 2; \left(\frac{1 - a}{b} \right)^2 r^2 \right) \quad \text{for } 0 < r \leq 1,$$

where $s_1 = (b^2 - a^2 + 2a - 1)/b$ and $s_2 = (b^2 - a^2 + a)/(1 - a)$; $a + b \geq 1$, $a \in [b, 1 + b]$. The maximum is attained by the rotation of the function $k_{(b^2 - a^2 + a)/b, (1 - a)/b}(z)$ defined by (1.1).

In Section 3, we present useful lemmas which are the main tools to prove our main theorems.

3. PREPARATORY RESULTS

If $f \in \mathcal{A}$ such that $z/f(z)$ is non-vanishing in \mathbb{D} (eg. the non-vanishing condition is ensured whenever $f \in \mathcal{S}$), then

$$(3.1) \quad \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

We first present a necessary coefficient condition for a function f of the form (3.1) to be in $\mathcal{S}^*(A, B)$.

Lemma 3.1. *Let $f \in \mathcal{S}^*(A, B)$ for $-1 \leq B \leq 0$ and $A \neq B$ and f be of the form (3.1). Then*

$$\sum_{k=1}^{\infty} (k^2 - |B - A - kB|^2) |b_k|^2 \leq |A - B|^2$$

holds.

Proof. Denote by $g(z) := z/f(z)$, $f \in \mathcal{S}^*(A, B)$. Then g has the form (3.1) and satisfies the relation

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)} \prec 1 - \frac{1 + Az}{1 + Bz} = \frac{(B - A)z}{1 + Bz}, \quad z \in \mathbb{D}.$$

Then by the definition of subordination, there exists an analytic function $w : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ with $w(0) = 1$ such that

$$\frac{zg'(z)}{g(z)} = \frac{(B - A)zw(z)}{1 + Bzw(z)}, \quad z \in \mathbb{D}.$$

Writing this in series form, we get

$$\sum_{k=1}^{\infty} kb_k z^{k-1} = \left((B - A) + \sum_{k=1}^{\infty} (B - A - kB)b_k z^k \right) w(z);$$

or equivalently

$$\sum_{k=1}^n kb_k z^{k-1} + \sum_{k=n+1}^{\infty} c_k z^{k-1} = \left((B - A) + \sum_{k=1}^{n-1} (B - A - kB)b_k z^k \right) w(z)$$

for certain coefficients c_k . Since $|w(z)| < 1$ in \mathbb{D} , Parseval-Gutzmer formula (see also Clunie's method [1] and [2, 14, 15]), we obtain

$$\sum_{k=1}^n k^2 |b_k|^2 r^{2k-2} \leq |A - B|^2 + \sum_{k=1}^{n-1} |B - A - kB|^2 |b_k|^2 r^{2k}.$$

or equivalently,

$$(3.2) \quad \sum_{k=1}^n k^2 |b_k|^2 r^{2k-2} - \sum_{k=1}^{n-1} |B - A - kB|^2 |b_k|^2 r^{2k} \leq |A - B|^2.$$

If we take $r = 1$ and allow $n \rightarrow \infty$, then we obtain the desired inequality

$$\sum_{k=1}^{\infty} (k^2 - |B - A - kB|^2) |b_k|^2 \leq |A - B|^2.$$

This completes the proof of our lemma. \square

Lemma 3.2. *Let $0 < |A| \leq 1$ and $f \in \mathcal{S}^*(A, 0)$. For $|z| < r$, suppose that*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k \text{ and } e^{-Az} = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad r \in (0, 1].$$

Then

$$(3.3) \quad \sum_{k=1}^N k |b_k|^2 r^{2k} \leq \sum_{k=1}^N k |c_k|^2 r^{2k}$$

holds for each $N \in \mathbb{N}$.

Proof. Clearly, it suffices to prove the lemma for $0 < A \leq 1$. From Lemma 3.1, using the equation (3.2) for $B = 0$, and then multiplying the resulting equation by r^2 on both sides shows that

$$(3.4) \quad \sum_{k=1}^{n-1} (k^2 - A^2 r^2) |b_k|^2 r^{2k} + n^2 |b_n|^2 r^{2n} \leq A^2 r^2.$$

The function e^{Az} clearly shows that the equality, when $n \rightarrow \infty$, in (3.4) attains with $b_k = c_k$.

Step-I: Cramer's Rule.

We consider the inequalities corresponding to (3.4) for $n = 1, \dots, N$ and multiply the n th coefficient by a factor $\lambda_{n,N}$. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (3.3). For the calculation of the factors $\lambda_{n,N}$ we get the following system of linear equations

$$(3.5) \quad k = k^2 \lambda_{k,N} + \sum_{n=k+1}^N \lambda_{n,N} (k^2 - A^2 r^2), \quad k = 1, \dots, N.$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (3.5) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2} \text{Det } A_{n,N},$$

where $A_{n,N}$ is the $(N - n + 1) \times (N - n + 1)$ matrix constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2 - A^2 r^2 & \dots & n^2 - A^2 r^2 \\ n+1 & (n+1)^2 & \dots & (n+1)^2 - A^2 r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \dots & N^2 \end{bmatrix}.$$

Determinants of these matrices can be obtained by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2 . The rest

of the entries are zeros. This expansion and a mathematical induction results in the following formula. If $k \leq N - 1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N} \left(1 - \frac{A^2 r^2}{k^2} \right) \prod_{m=k+1}^{N-1} \left(\frac{A^2 r^2}{m^2} \right).$$

For fixed $k \in \mathbb{N}$ and $N \geq k$, we see that the sequence $\{\lambda_{k,N}\}$ is strictly non-increasing, i.e. $\lambda_{k,N} - \lambda_{k,N-1} < 0$ with

$$(3.6) \quad \lambda_k := \lim_{N \rightarrow \infty} \lambda_{k,N} = \frac{1}{k} - \left(1 - \frac{A^2 r^2}{k^2} \right) \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\frac{A^2 r^2}{m^2} \right).$$

To prove that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}, 1 \leq k \leq N$, it is adequate to show that $\lambda_k \geq 0$ for $k \in \mathbb{N}$. This will be completed in Step II. But before that we want to remark that the proof of the said inequality is sufficient for the proof of the theorem, since, as we remarked for (3.4), equality holds for $b_k = c_k$.

Step-II: Positivity of the Multipliers.

Let for an abbreviation

$$S_k = \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\frac{A^2 r^2}{m^2} \right), \quad k \in \mathbb{N}.$$

We now show that

$$S_k \leq \frac{1}{k \left(1 - \frac{A^2 r^2}{k^2} \right)}.$$

From the relation (3.6), we have

$$\lambda_k = \frac{1}{k} - S_k + \left(\frac{A^2 r^2}{k^2} \right) S_k.$$

Again set for an abbreviation

$$T_k = \frac{1}{k} + \left(\frac{A^2 r^2}{k^2} \right) S_k.$$

It is enough to show that

$$(3.7) \quad T_k \leq \frac{1}{k \left(1 - \frac{A^2 r^2}{k^2} \right)}.$$

To show (3.7) we use the inequality

$$(3.8) \quad \frac{1}{n \left(1 - \frac{A^2 r^2}{n^2} \right)} > \frac{1}{(n+1) \left(1 - \frac{A^2 r^2}{(n+1)^2} \right)}$$

and the identity

$$(3.9) \quad \frac{1}{n \left(1 - \frac{A^2 r^2}{n^2} \right)} = \frac{1}{n} + \frac{\frac{A^2 r^2}{n^2}}{n \left(1 - \frac{A^2 r^2}{n^2} \right)}$$

which are admissible for each $n \in \mathbb{N}$. Repeated application of (3.8) and (3.9) for $n = k, k+1, \dots, P$ results the inequality

$$\frac{1}{k \left(1 - \frac{A^2 r^2}{k^2}\right)} > \sum_{n=k}^P \frac{1}{n} \prod_{m=k}^{n-1} \left(\frac{A^2 r^2}{m^2} \right) + \frac{\prod_{m=k}^P \left(\frac{A^2 r^2}{m^2} \right)}{P \left(1 - \frac{A^2 r^2}{P^2}\right)} =: S_{k,P} + R_{k,P}, \text{ for } k \leq P.$$

Since $R_{k,P} > 0$ and taking limit $P \rightarrow \infty$, we obtain

$$\frac{1}{k \left(1 - \frac{A^2 r^2}{k^2}\right)} \geq \lim_{P \rightarrow \infty} S_{k,P} = \sum_{n=k}^{\infty} \frac{1}{n} \prod_{m=k}^{n-1} \left(\frac{A^2 r^2}{m^2} \right).$$

Hence, we get the relation (3.7). The proof of our lemma is complete. \square

Lemma 3.3. *Let $-1 \leq B < 0$, $A \neq B$ and $f \in \mathcal{S}^*(A, B)$. For $|z| < r$, suppose that*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k \text{ and } (1 - Bz)^{1-A/B} = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad r \in (0, 1].$$

Then

$$(3.10) \quad \sum_{k=1}^N k |b_k|^2 r^{2k} \leq \sum_{k=1}^N k |c_k|^2 r^{2k}, \quad N \in \mathbb{N}$$

is recognized.

Proof. As in the proof of Lemma 3.2, we may rewrite (3.2) in the form

$$(3.11) \quad \sum_{k=1}^{n-1} (k^2 - |k - \phi|^2 B^2 r^2) |b_k|^2 r^{2k} + n^2 |b_n|^2 r^{2n} \leq B^2 |\phi|^2 r^2,$$

where $\phi := 1 - A/B$. The function $(1 - Bz)^{1-A/B}$ clearly shows that the equality, when $n \rightarrow \infty$, in (3.11) attains with $b_k = c_k$.

Rest of the proof is divided into two steps.

Step-I: Cramer's Rule.

We consider the inequalities corresponding to (3.11) for $n = 1, \dots, N$ and multiply the n th coefficient by a factor $\lambda_{n,N}$. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (3.10). For the calculation of the factors $\lambda_{n,N}$ we get the following system of linear equations

$$(3.12) \quad k = k^2 \lambda_{k,N} + \sum_{n=k+1}^N \lambda_{n,N} (k^2 - |k - \phi|^2 B^2 r^2), \quad k = 1, \dots, N.$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (3.12) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2} \text{Det } A_{n,N},$$

where $A_{n,N}$ is the $(N - n + 1) \times (N - n + 1)$ matrix constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2 - |n - \phi|^2 B^2 r^2 & \cdots & n^2 - |n - \phi|^2 B^2 r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2 - |n+1 - \phi|^2 B^2 r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \cdots & N^2 \end{bmatrix}.$$

Determinants of these matrices can be found by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2 . The rest of the entries are zeros. This expansion and a mathematical induction results in the following formula. If $k \leq N - 1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N} \left(1 - \left| 1 - \frac{\phi}{k} \right|^2 B^2 r^2 \right) \prod_{m=k+1}^{N-1} \left| 1 - \frac{\phi}{m} \right|^2 B^2 r^2.$$

Set as an abbreviation $U_k = 1 - |1 - \phi/k|^2 B^2 r^2$, we get

$$(3.13) \quad \lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N} U_k \prod_{m=k+1}^{N-1} (1 - U_m).$$

Note that U_k in (3.13) may be positive as well as negative for all $k \in \mathbb{N}$. We investigate it by including here a table (see Table 3).

k	A	B	r	U_k
1	$2 + i$	all	0.5	-5.25
1	$1 + i$	all	0.4	0.2
2	$-2 + i$	-1	0.5	0.375
2	$-2 + i$	-1	0.8	-0.6

Table 3

Case (i): Suppose that U_k is negative.

From the relation (3.13), we see that for fixed $k \in \mathbb{N}, k \leq N - 1$, the sequence $\{\lambda_{k,N}\}$ is strictly non-decreasing, i.e.

$$\lambda_{k,N} - \lambda_{k,N-1} > 0$$

so that

$$\lambda_{k,N} > \lambda_{k,N-1} > \cdots > \lambda_{k,k} = 1/k > 0,$$

and thus $\lambda_k \geq 0$ when $N \rightarrow \infty$ as required.

Case (ii): Suppose that U_k is positive.

For fixed $k \in \mathbb{N}, N \geq k$, the sequence $\{\lambda_{k,N}\}$ is strictly non-increasing, i.e. $\lambda_{k,N} - \lambda_{k,N-1} < 0$ with

$$(3.14) \quad \lambda_k := \lim_{N \rightarrow \infty} \lambda_{k,N} = \frac{1}{k} - U_k \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} (1 - U_m).$$

For all $N \in \mathbb{N}, 1 \leq k \leq N$, to prove that $\lambda_{k,N} > 0$, it is sufficient to prove $\lambda_k \geq 0$ for $k \in \mathbb{N}$. This will be completed in Step II. But before that we want to annotate that the proof of the said inequality is sufficient for the proof of the theorem, since, as we noted in the beginning of the proof, equality is received for $b_k = c_k$.

Step-II: Positivity of the Multipliers.

Let for an abbreviation

$$S_k = \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} (1 - U_m), \quad k \in \mathbb{N}.$$

We now prove that

$$S_k \leq \frac{1}{kU_k}.$$

From the relation (3.14), we get

$$\lambda_k = \frac{1}{k} - S_k + (1 - U_k)S_k.$$

Again set for an abbreviation

$$T_k = \frac{1}{k} + (1 - U_k)S_k.$$

It is enough to prove that

$$(3.15) \quad T_k \leq \frac{1}{kU_k}.$$

To prove (3.15) we use the inequality

$$(3.16) \quad \frac{1}{nU_n} > \frac{1}{(n+1)U_{n+1}}$$

and the identity

$$(3.17) \quad \frac{1}{nU_n} = \frac{1}{n} + \frac{1 - U_n}{nU_n}$$

which are valid for each $n \in \mathbb{N}$. Repeated application of (3.16) and (3.17) for $n = k, k+1, \dots, P$ results the inequality

$$\frac{1}{kU_k} > \sum_{n=k}^P \frac{1}{n} \prod_{m=k}^{n-1} (1 - U_m) + \frac{\prod_{m=k}^P (1 - U_m)}{PU_P} =: S_{k,P} + R_{k,P}, \quad \text{for } k \leq P.$$

Since $R_{k,P} > 0$, taking the limit as $P \rightarrow \infty$ we obtain

$$\frac{1}{kU_k} \geq \lim_{P \rightarrow \infty} S_{k,P} = \sum_{n=k}^{\infty} \frac{1}{n} \prod_{m=k}^{n-1} (1 - U_m),$$

and we complete the inequality (3.15). This completes the proof of Lemma 3.3. \square

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 2.1. Let $f \in \mathcal{S}^*(A, 0)$. By the definition of the class $\mathcal{S}^*(A, 0)$, it suffices to assume that $0 < A \leq 1$ and

$$\frac{zf'(z)}{f(z)} \prec 1 - Az = \frac{zk'_{A,0}(z)}{k_{A,0}(z)}, \quad z \in \mathbb{D}.$$

By the subordination principle, we obtain that $z/f(z) \prec e^{-Az}$ which in terms of the Taylor coefficients may be written as

$$1 + \sum_{n=1}^{\infty} b_n z^n \prec e^{-Az} = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad c_n = (-1)^n \frac{A^n}{n!}.$$

By Lemma 3.2, we have

$$\sum_{n=1}^N n |b_n|^2 r^{2n} \leq \sum_{n=1}^N n |c_n|^2 r^{2n}, \quad N \in \mathbb{N}, \quad r \in (0, 1),$$

which implies that

$$\Delta \left(r, \frac{z}{f} \right) = \pi \sum_{n=1}^{\infty} n |b_n|^2 r^{2n} \leq \Delta \left(r, \frac{z}{k_{A,0}} \right) = \pi \sum_{n=1}^{\infty} n |c_n|^2 r^{2n}.$$

We claim that

$$\pi \sum_{n=1}^{\infty} n |c_n|^2 r^{2n} = E_{A,0}(r),$$

where $E_{A,0}(r) = \pi A^2 r^2 {}_0F_1(2; A^2 r^2)$ with $0 < A \leq 1$. To prove the claim, we observe that

$$\begin{aligned} \pi^{-1} \Delta \left(r, \frac{z}{k_{A,0}} \right) &= \sum_{n=1}^{\infty} n \frac{A^{2n}}{(n!)^2} r^{2n} \\ &= A^2 r^2 \sum_{n=0}^{\infty} \frac{1}{(2)_n (1)_n} A^{2n} r^{2n} \\ &= A^2 r^2 {}_0F_1(2; A^2 r^2) \\ &:= \pi^{-1} E_{A,0}(r) \end{aligned}$$

and thus,

$$\Delta \left(r, \frac{z}{f} \right) \leq \Delta \left(r, \frac{z}{k_{A,0}} \right) = E_{A,0}(r).$$

The equality case is obvious from $z/k_{A,0}(z) = e^{-Az}$. The proof of the theorem is complete. \square

Proof of Theorem 2.3. Suppose $f \in \mathcal{S}^*(A, B)$, $-1 \leq B < 0$ and $A \neq B$. Then by setting $g(z) := z/f(z)$, we write

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)} \prec 1 - \frac{1 + Az}{1 + Bz} = \frac{(B - A)z}{1 + Bz}, \quad z \in \mathbb{D}.$$

By a well-known subordination result, we get

$$g(z) = \frac{z}{f(z)} \prec (1 + Bz)^{1-\frac{A}{B}} = \frac{z}{k_{A,B}(z)},$$

where $k_{A,B}$ is defined by (1.1). If

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \quad \text{and} \quad \frac{z}{k_{A,B}(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad |z| < r,$$

then Lemma 3.3 gives that

$$\sum_{n=1}^N n|b_n|^2 r^{2n} \leq \sum_{n=1}^N n|c_n|^2 r^{2n}$$

for each $N \in \mathbb{N}$ and $r \in (0, 1]$. Allowing $N \rightarrow \infty$, we obtain

$$\Delta\left(r, \frac{z}{f}\right) = \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \leq \Delta\left(r, \frac{z}{k_{A,B}}\right) = \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

Clearly,

$$c_n = (-1)^n \frac{(\zeta)_n}{(1)_n} B^n \quad \text{with } \zeta = (A/B) - 1.$$

Now, applying the area formula (1.3) for the function $z/k_{A,B}(z)$, we obtain that

$$\begin{aligned} \pi^{-1} \Delta\left(r, \frac{z}{k_{A,B}}\right) &= \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}, \quad |z| < r \\ &= \sum_{n=1}^{\infty} n \frac{(\zeta)_n (\bar{\zeta})_n}{(1)_n (1)_n} B^{2n} r^{2n} \\ &= |\zeta|^2 B^2 r^2 \sum_{n=0}^{\infty} \frac{(\zeta+1)_n (\bar{\zeta}+1)_n}{(2)_n (1)_n} B^{2n} r^{2n} \\ &= |\bar{A} - B|^2 r^2 {}_2F_1(A/B, \bar{A}/B; 2; B^2 r^2) \\ &:= \pi^{-1} E_{A,B}(r), \end{aligned}$$

and the proof of Theorem 2.3 is complete. \square

5. CONCLUDING REMARKS

It would be interesting to solve the analog of Yamashita's extremal problem (1.4) for many interesting geometric subclasses of functions from \mathcal{S} . For example, determine the analog of Theorems 2.1 and 2.3 when zf' belongs to the class $\mathcal{S}^*(A, B)$ and also for functions f in the Bazilevič class or for functions convex in some direction.

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REFERENCES

- [1] J. G. Cluine, On meromorphic schlicht functions, *J. London Math. Soc.*, **34**(1959), 215–216.
- [2] J. G. Clunie and F. R. Keogh, On starlike and convex schlicht functions, *J. London Math. Soc.*, **35**(1960), 229–233.
- [3] P. L. Duren, *Univalent Functions*, Springer-Verlag, 1983.
- [4] A. W. Goodman, *Univalent Functions*, Vol. 1-2, Mariner, Tampa, Florida, 1983.
- [5] W. Janowski, Some extremal problems for certain families of analytic functions, *Ann. Polon. Math.*, **28**(1981), 297–326.
- [6] R. J. Libera, Univalent α -spiral functions, *Canad. J. Math.*, **19**(1967), 449–456.

- [7] R. Nevalinna, Über die konforme Abbildung von Sterngebieten, *Öresikt av Finska Vetenskaps Soc. Förh.*, **63A**, no. 6(1921), 1–21.
- [8] M. Obradović, S. Ponnusamy, and K.-J. Wirths, A proof of Yamashita’s conjecture on area integral, *Comput. Methods Funct. Theory*, **13**(2013), 479–492.
- [9] M. Obradović, S. Ponnusamy, and K.-J. Wirths, Integral means and Dirichlet integral for analytic functions, *Mathematische Nachrichten* (2014), 9 pages, To appear.
- [10] K. S. Padmanabhan, On certain classes of starlike functions in the unit disk, *J. Indian Math. Soc.*, (N.S.) **32**(1968), 89–103.
- [11] S. Ponnusamy, and K.-J. Wirths, On the problem of Gromova and Vasil’ev on integral means, and Yamashita’s conjecture for spirallike functions, *Ann. Acad. Sci. Fenn. Ser. AI Math.* **39**(2014), 721–731.
- [12] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.
- [13] M. S. Robertson, On the theory of univalent functions, *Annals of Mathematics*, **37**(1936), 374–408.
- [14] M. S. Robertson, Quasi-subordination and coefficient conjectures, *J. Bull. Amer. Math. Soc.*, **76**(1970), 1–9.
- [15] W. Rogosinski, On the coefficients of subordinate functions, *Proc. London Math. Soc.*, **48**(2)(1943), 48–82.
- [16] H. Silverman, Subclass of starlike functions, *Rev. Roum. Math. Pure Appl.*, **33**(1978), 1093–1099.
- [17] R. Singh, On a class of starlike functions, *J. Indian Math. Soc.*, **32**(1968), 208–213.
- [18] R. Singh and V. Singh, On a class of bounded starlike functions, *Indian J. Pure Appl. Math.*, **5**(1974), 733–754.
- [19] S. K. Sahoo and N. L. Sharma, On area integral problem for analytic functions in the starlike family, preprint (see the preprint available at *arXiv:1405.0469 [math.CV]*).
- [20] L. Špaček, Contribution à la théorie des fonctions univalentes (in Czech), *Časop Pěst. Mat.-Fys.*, **62**(1933), 12–19.
- [21] S. Yamashita, Area and length maxima for univalent functions, *Proc. London Math. Soc.*, **41**(2)(1990), 435–439.

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